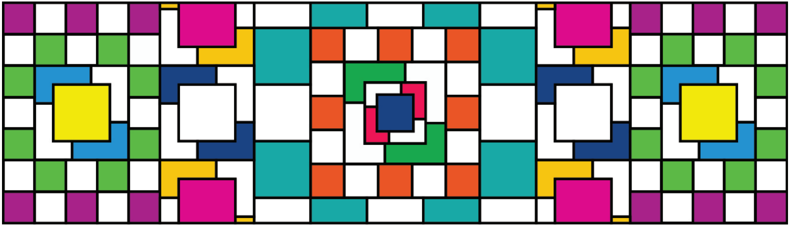
Patterns and Partitions

Yotam Smilansky

Experimental Mathematics Seminar, Rutgers University



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2. Color frequencies:
   - Does $|r_m| / |x_m|$ converge?
   - Does $\mathcal{L}(\{x \in \mathcal{I} : x \text{ is colored red in } \pi_m\})$ converge?
   - In case both limits exist, are they necessarily the same?

Multiscale substitution schemes

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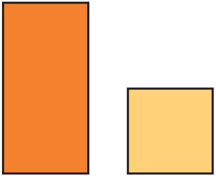

Prototiles in $\mathbb{R}^d$

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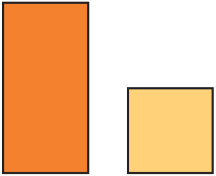

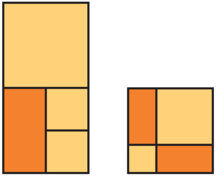

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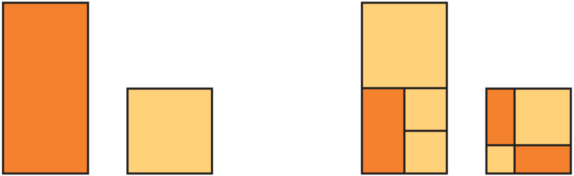

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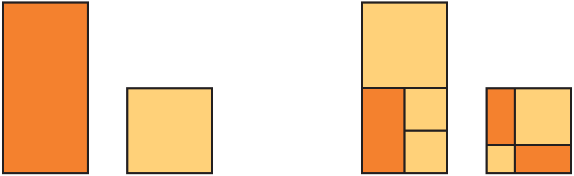

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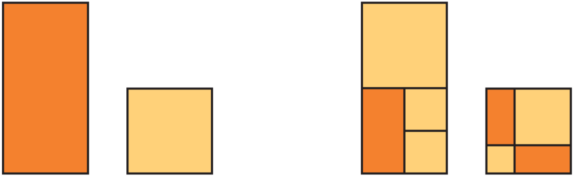

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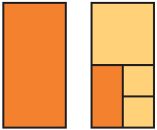

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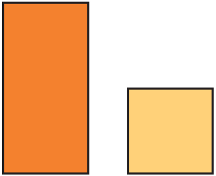

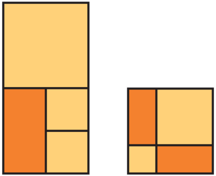

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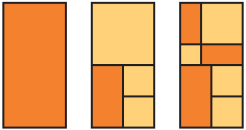

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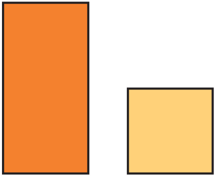

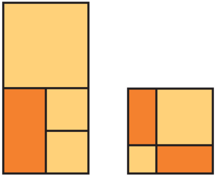

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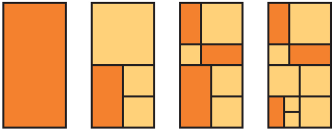

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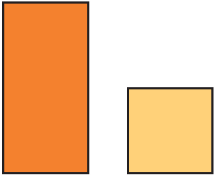

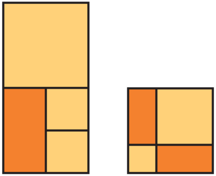

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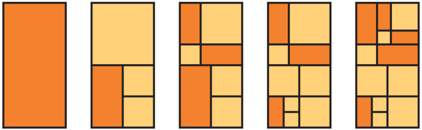

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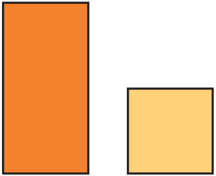

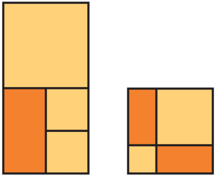

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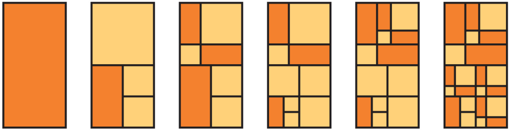

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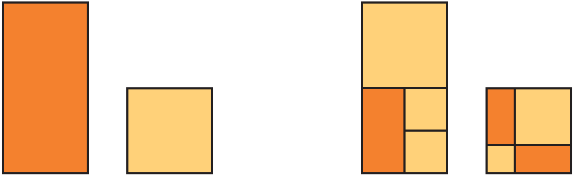

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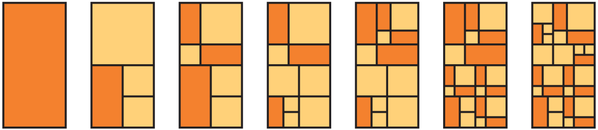

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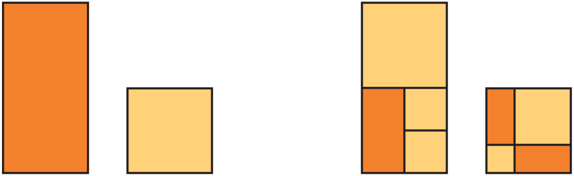

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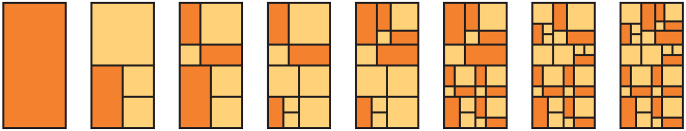

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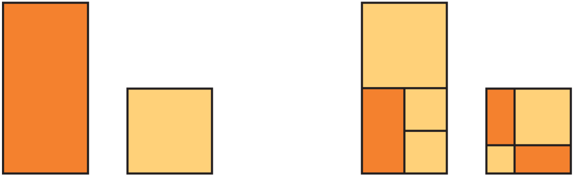

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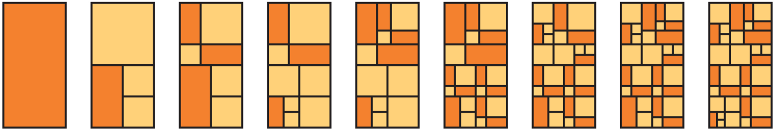
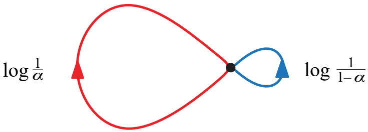


Directed weighted graph model for substitution schemes



Vertices model the prototiles.

Edges originating in a vertex model the tiles appearing in the substitution rule pattern of the prototile.

Lengths determined by the scales of the tiles.

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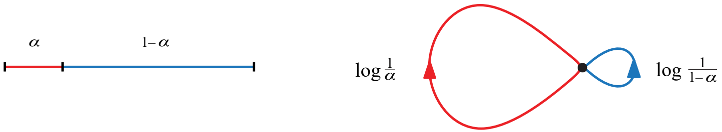

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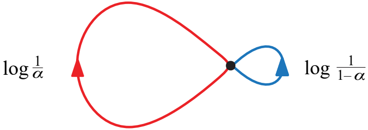

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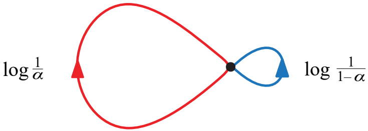

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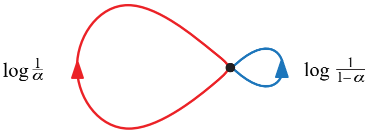

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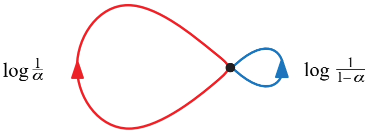

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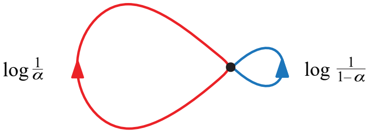

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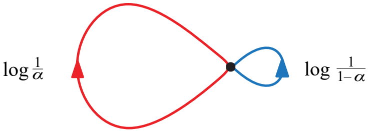

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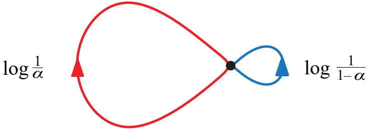

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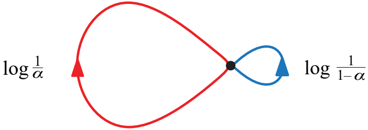

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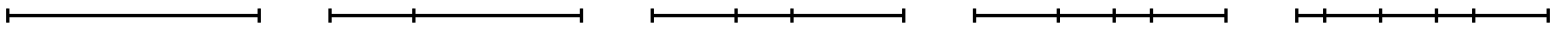

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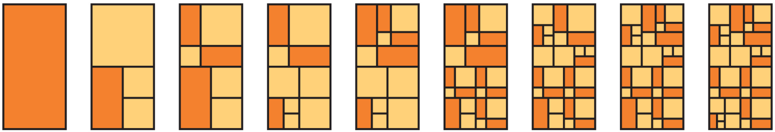

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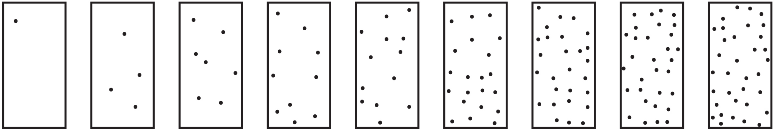

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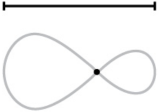

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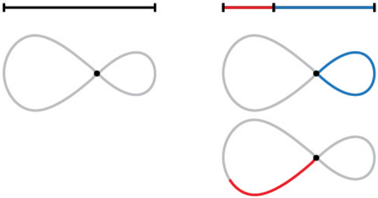

Splitting procedure and flow on edges

Metric paths are directed walks on edges, do not necessarily originate or terminate at vertices.

Tiles in partition π_m are modeled by metric paths of length l_m.

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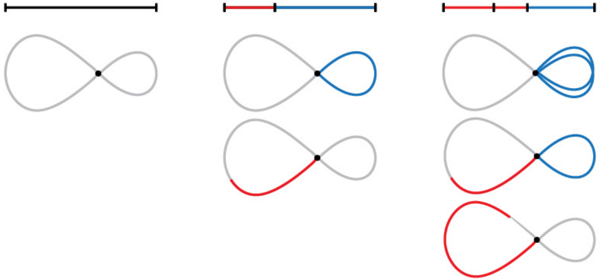

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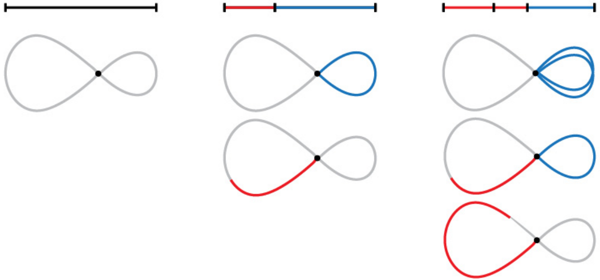

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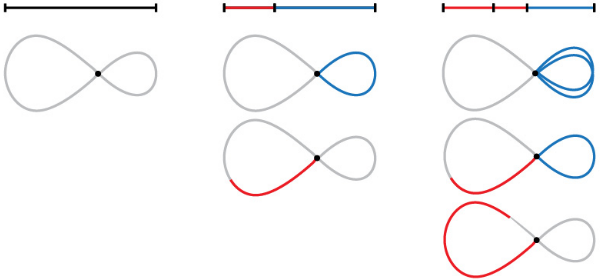

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$\mathcal{L}(\{x \in \mathcal{I} : x \text{ is colored red in } \pi_m\})$ is the probability that a metric path of length l_m terminates on the red edge, if the red edge is assigned probability $\frac{1}{3}$ and the blue edge probability $\frac{2}{3}$.

Incommensurable and commensurable schemes

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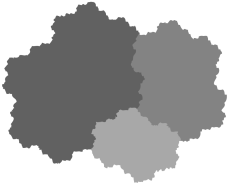
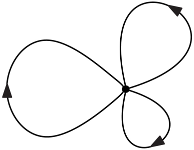
More examples

A commensurable example – **The Rauzy fractal** scheme:





Edge lengths: $\log \tau, 2\log \tau, 3\log \tau$, where $\tau =$ tribonacci constant.

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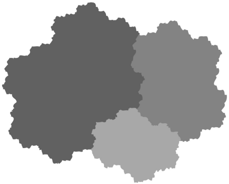

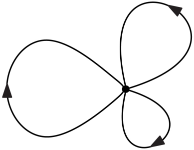

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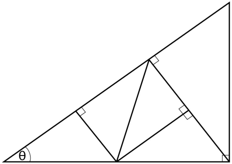
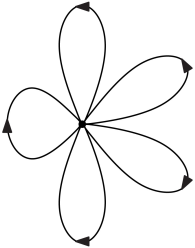
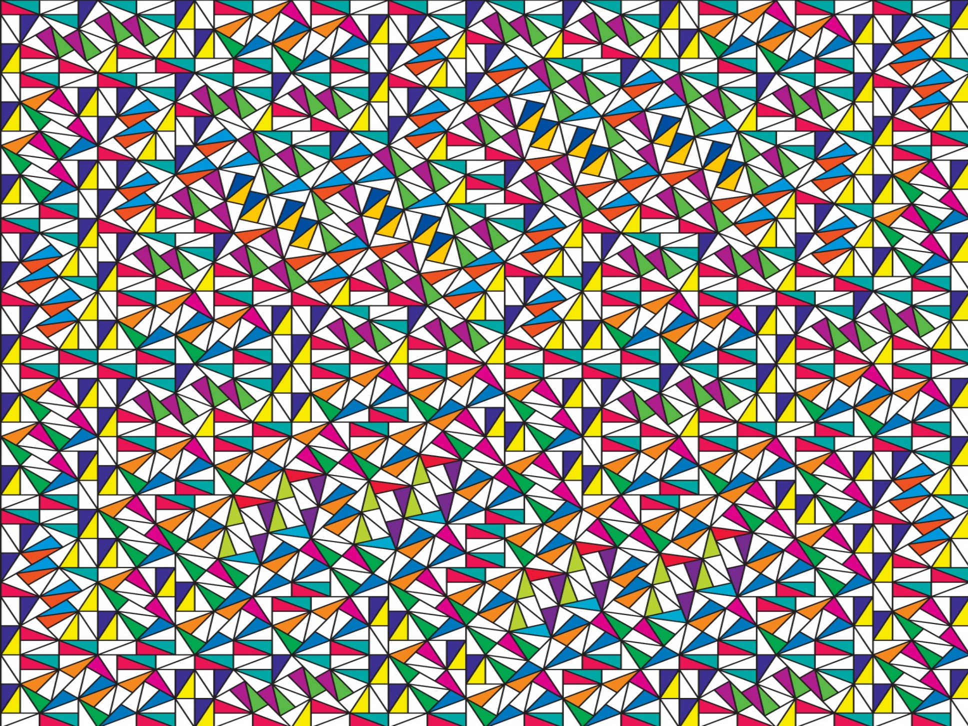
For a.e θ **Sadun's generalized pinwheel** scheme is incommensurable:







Multiscale substitution tilings (jointly with Yaar Solomon)

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Given am incommensurable scheme and starting with a prototile T of volume 1, the substitution flow $F_t(T)$ is defined by

- At $t = 0$ the tile T is substituted.
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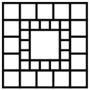
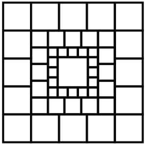
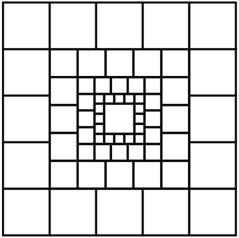
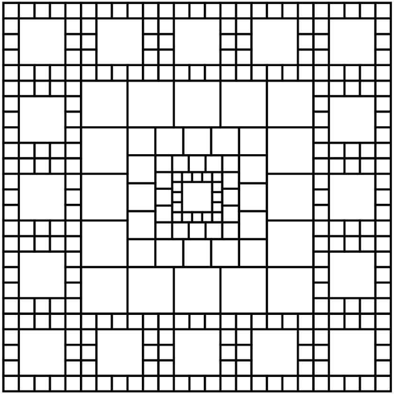
Our study includes:

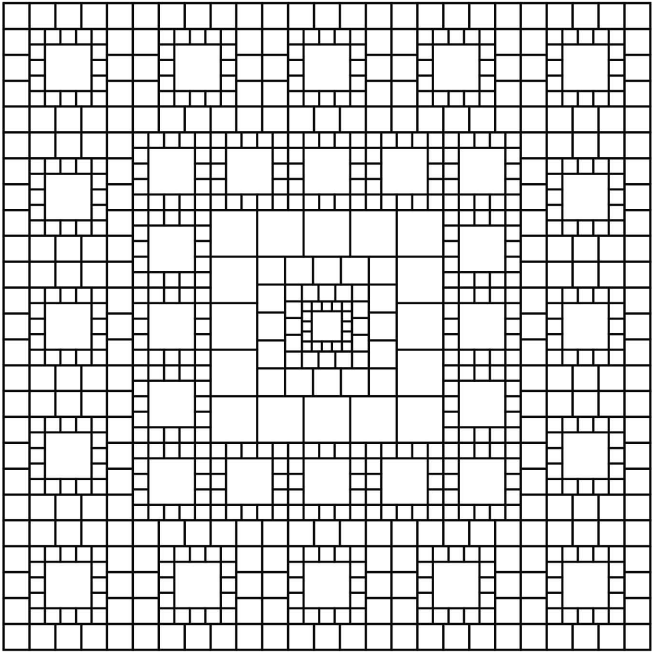
- Structural, geometrical and statistical properties of tilings: (types and scales, repetitivity, patch frequencies, BD/BL)
- Dynamical properties of the **tiling dynamical system**. (minimality, invariant measures)

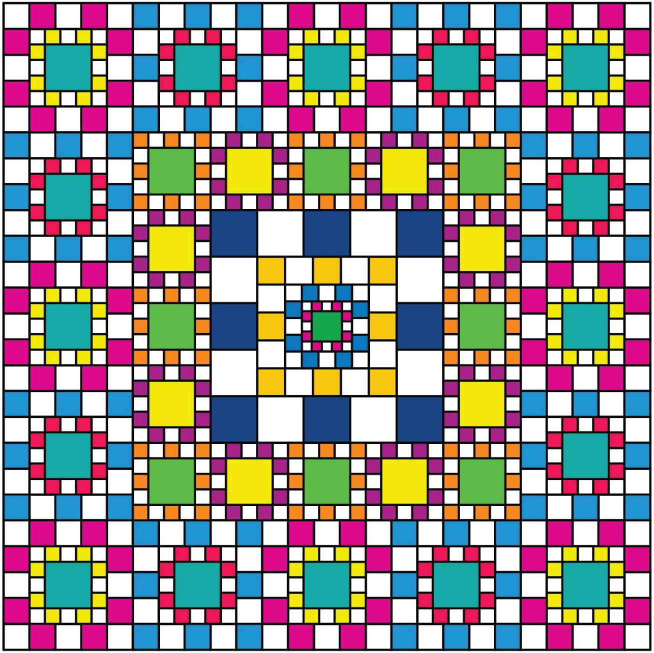






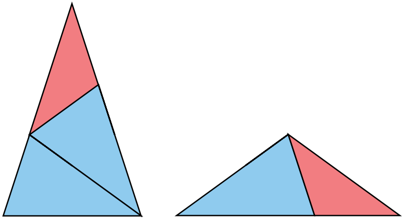






The commensurable case

Famous constructions such as the Penrose-Robinson scheme:



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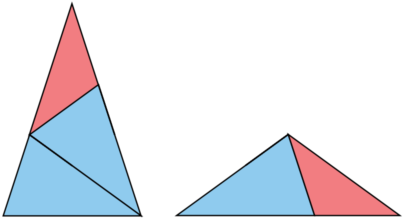

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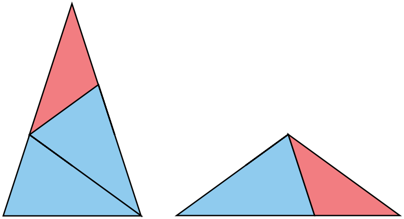

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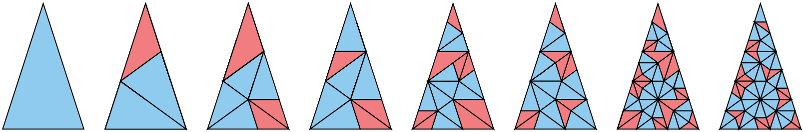

This Kakutani sequence **does not** have color frequencies.

The incommensurable case - counting paths on graphs

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Let $M(s)$ be the **graph matrix function** defined by

$$M_{ij}(s) = e^{-s \cdot l(\varepsilon_1)} + \cdots + e^{-s \cdot l\left(\varepsilon_{k_{ij}}\right)},$$

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Theorem (Kiro, Smilansky$\times$2): Let G be a strongly connected incommensurable graph. There exist $\lambda > 0$ and $Q \in M_n(\mathbb{R})$ with positive entries, such that if $\varepsilon \in \mathcal{E}$ has initial vertex $h \in \mathcal{V}$, the number of metric paths of length exactly x from vertex $i \in \mathcal{V}$ to a point on the edge ε grows as

$$\frac{1 - e^{-l(\varepsilon)\lambda}}{\lambda} Q_{ih} e^{\lambda x} + o\left(e^{\lambda x}\right), \quad x \to \infty.$$

where λ is the maximal real value for which $\rho(M(\lambda)) = 1$,

$$Q = \frac{\mathrm{adj}(I - M(\lambda))}{-\mathrm{tr}\left(\mathrm{adj}(I - M(\lambda)) \cdot M'(\lambda)\right)}.$$

Poles of the Laplace transform

The proof follows **The Wiener-Ikehara Theorem**, originally motivated by the Prime Number Theorem.

This requires the study of the poles of the Laplace transform of a counting function, which in our case is given by

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Information on the location of zeroes closest to $Re(s) = \lambda$ can be used to obtain upper bounds on error terms.

Zeroes of exponential polynomial (jointly with Avner Kiro, Alon Nishry and Aron Wennman)

In the case of graphs modeling an α-Kakutani scheme

$$\det(I - M(s)) = 1 - e^{-as} - e^{-bs}$$

with $a = \log \frac{1}{\alpha}$ and $b = \log \frac{1}{1-\alpha}$.

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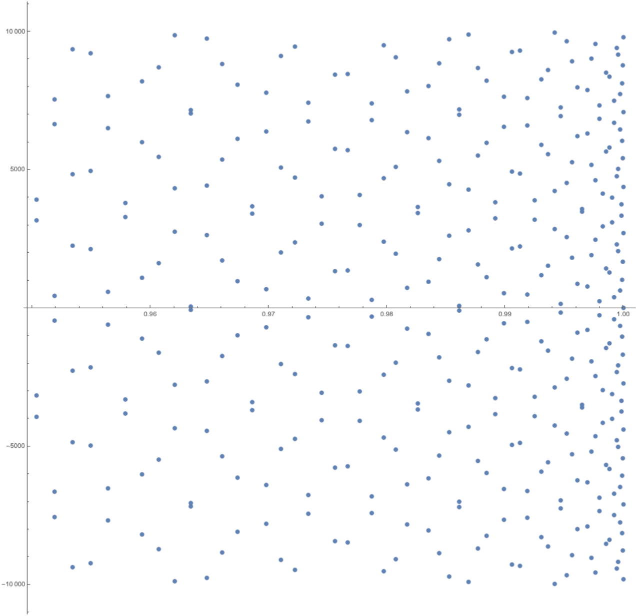
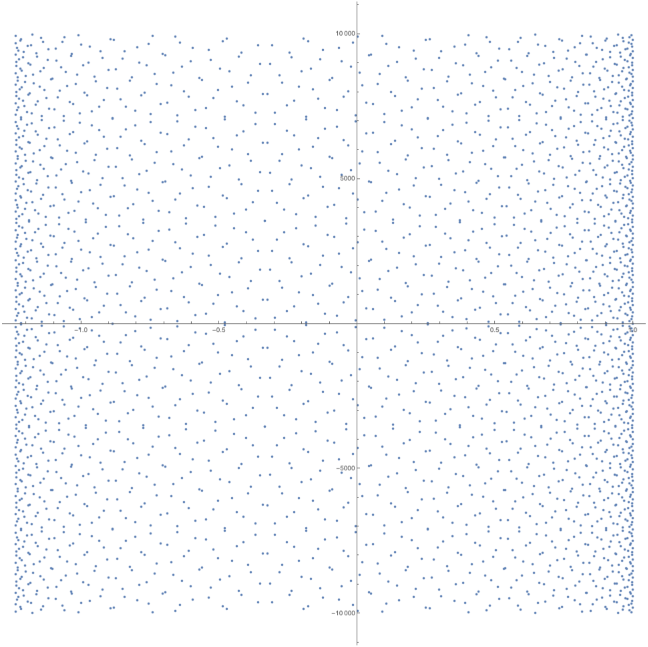
The following slides show some approximations of such zeroes in compact strips, for different values of β. At the moment these experimentations give rise to more questions than answers...

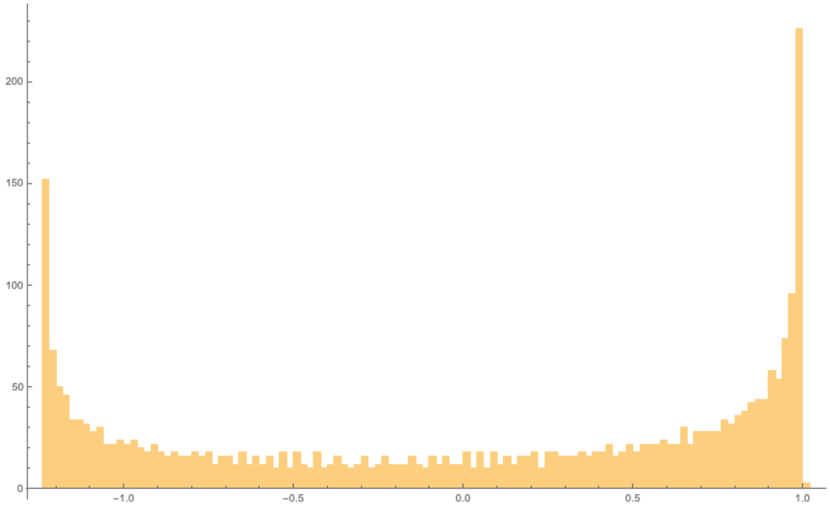
$\beta = \varphi$ the golden ratio, rightmost roots (up to 10,000)



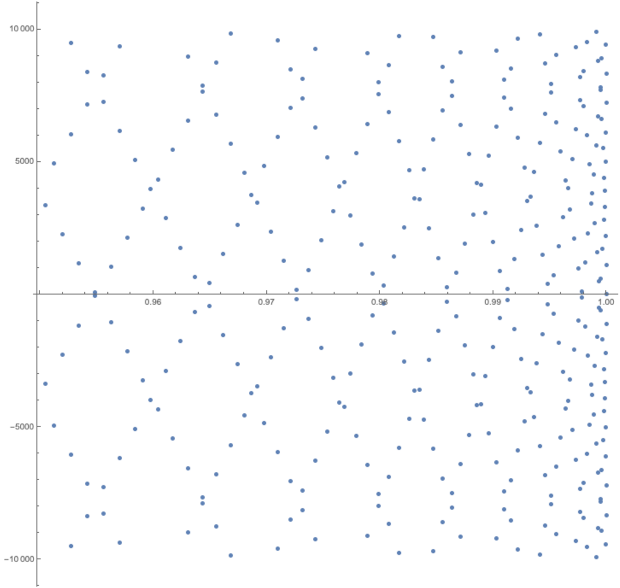
$\beta = \varphi$, all roots (up to 10,000)



$\beta = \varphi$, histogram



$\beta = e$, rightmost roots (up to 10,000)



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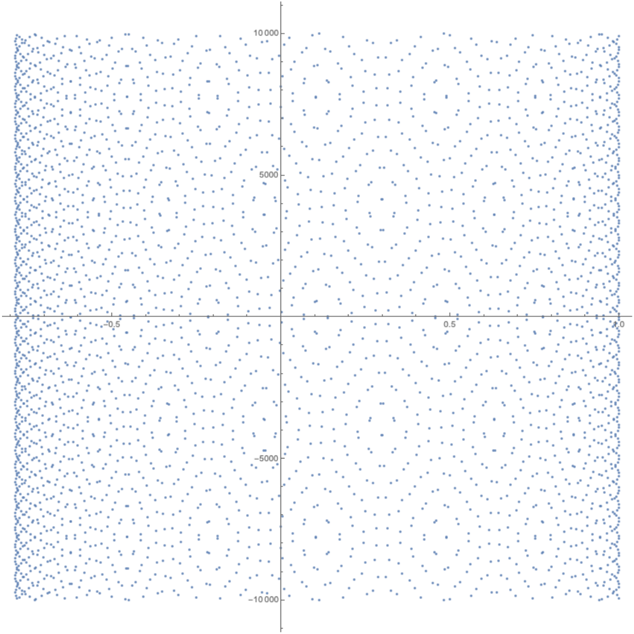

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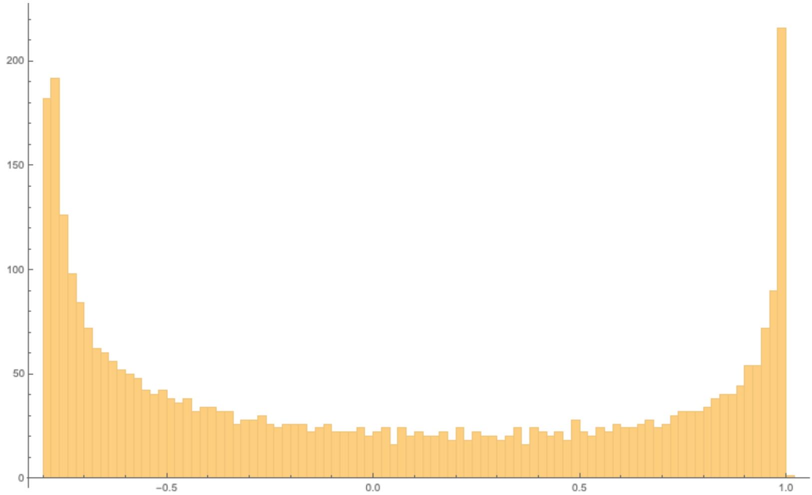
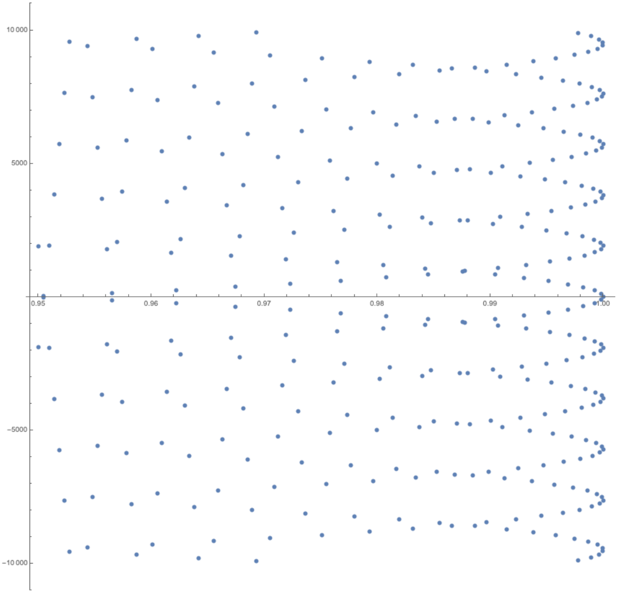


$\beta = \pi$, rightmost roots (up to 10,000)



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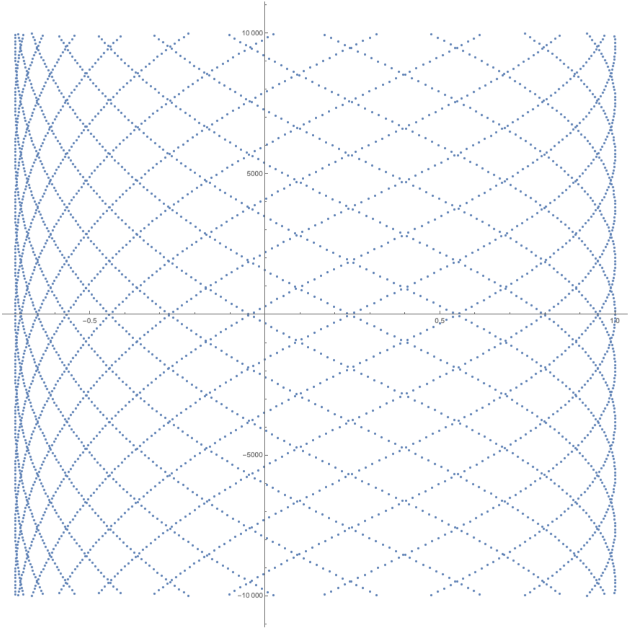

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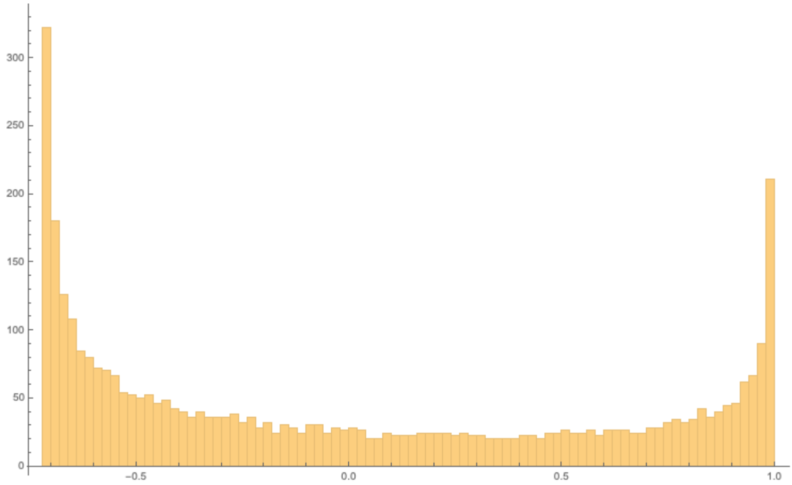
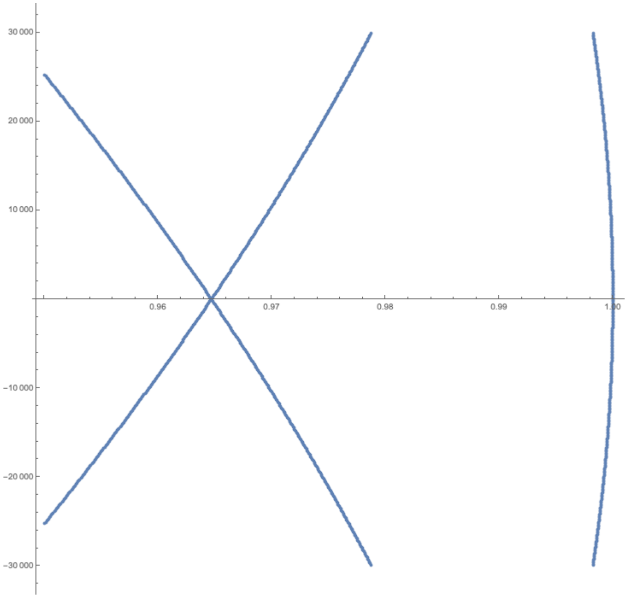


$\beta = \ell$, a Liouville number, rightmost roots (up to 30,000)



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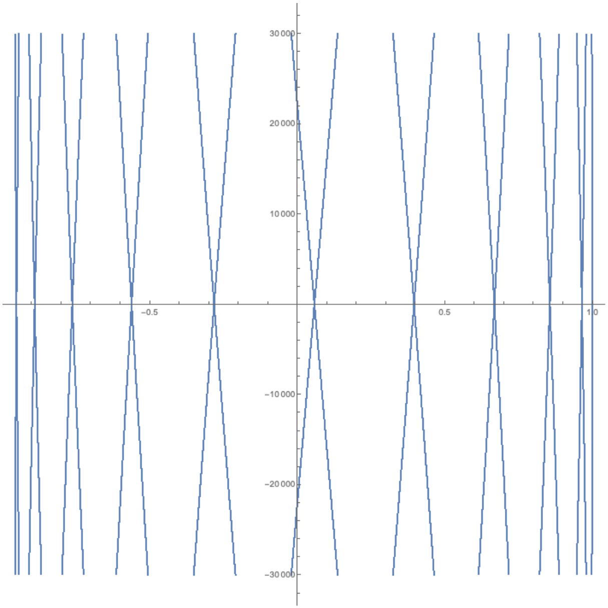

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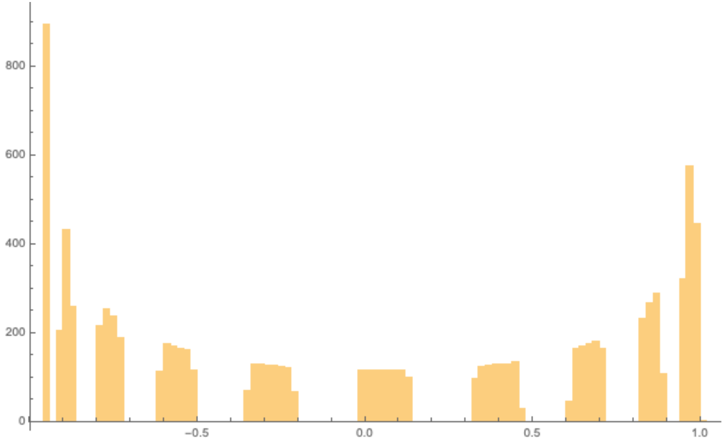


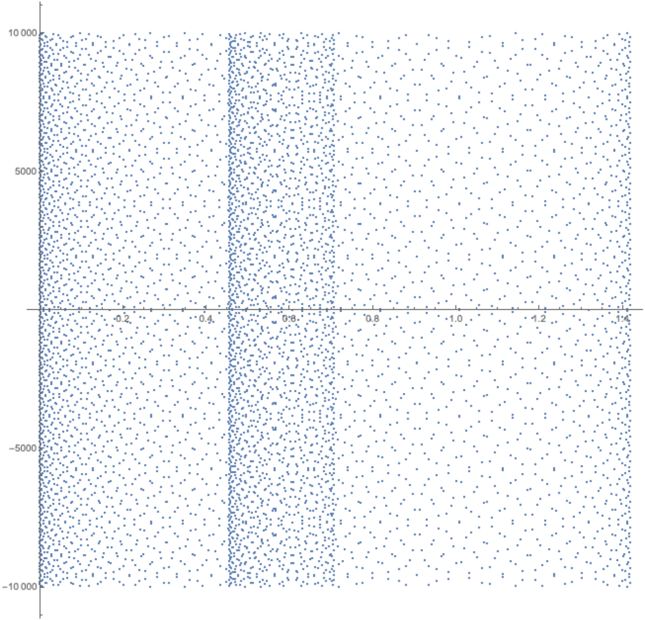
Extending the model

Next, we now turn to the roots of

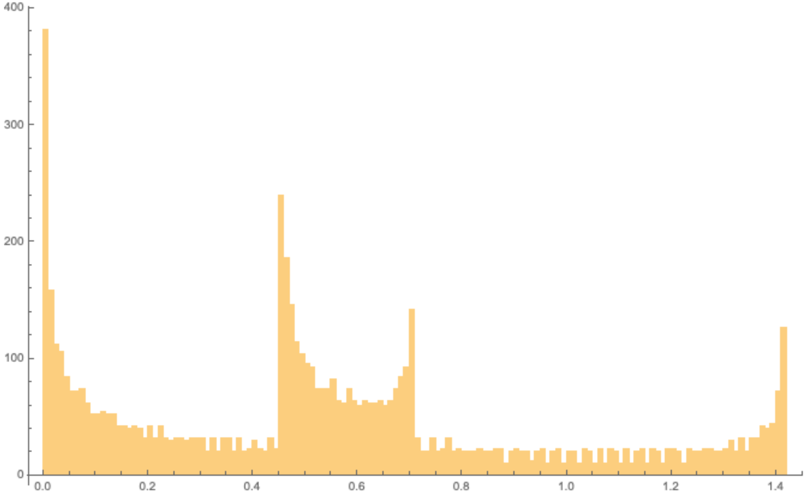
$$e^{z} + e^{\beta z} + e^{\gamma z} = 1,$$

which are related to graph with a vertex and three loops, or to schemes in which $\mathcal{I}$ is substituted by three rescaled copy of itself.

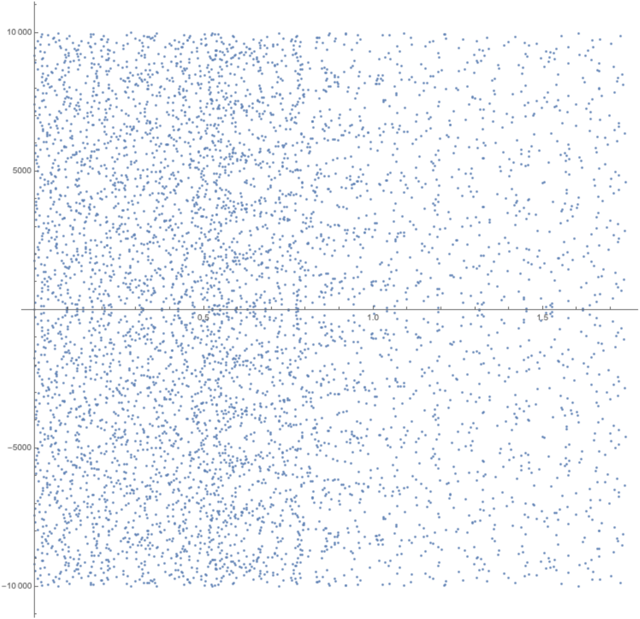
$\beta = 2$ and $\gamma = \varphi$, all roots (up to 10,000)



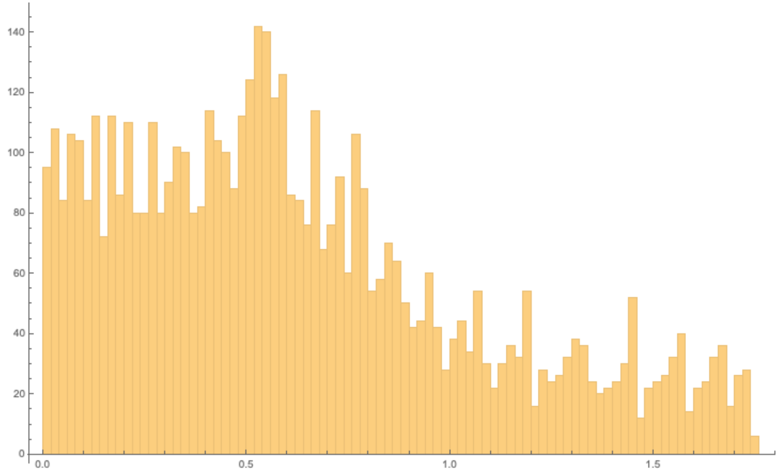
$\beta = 2$ and $\gamma = \varphi$, histogram



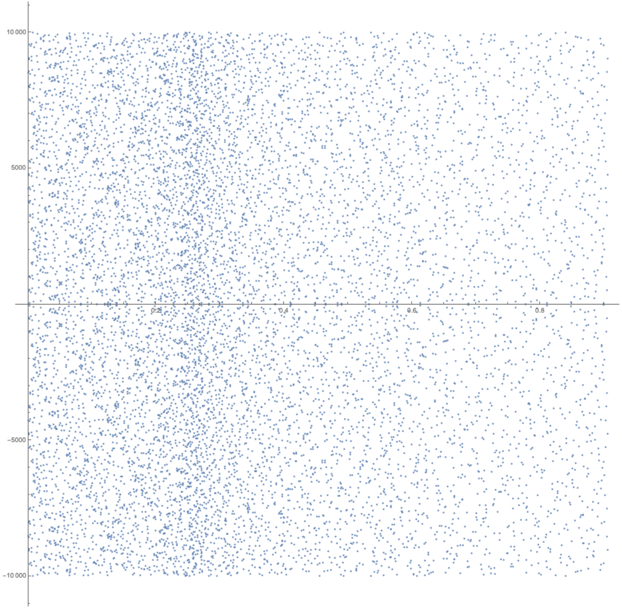
$\beta = \sqrt{2}$ and $\gamma = \sqrt{3}$, all roots (up to 10,000)



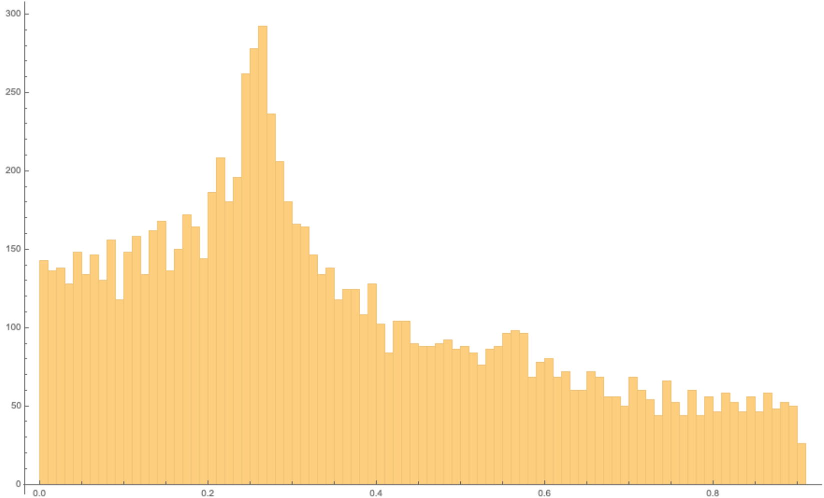
$\beta = \sqrt{2}$ and $\gamma = \sqrt{3}$, histogram



$\beta = e$ and $\gamma = \pi$, all roots (up to 10,000)



$\beta = e$ and $\gamma = \pi$, histogram



Thanks!